Collision Induced Decays of Electroweak Solitons: Fermion Number Violation with Two Initial Particles [1]

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Abstract

This paper presents work done in collaboration with E. Farhi, J. Goldstone, and A. Lue which is described in full in Ref. [1].

We consider a variant of the standard electroweak theory in which the Higgs sector has been modified so that there is a classically stable weak scale soliton. We explore fermion number violating processes which involve soliton decay. A soliton can decay by tunnelling under the sphaleron barrier, or the decay can be collision induced if the energy is sufficient for the barrier to be traversed. We discuss classical solutions to the Minkowski space equations of motion in which a soliton is kicked over the barrier by an incoming pulse. We then consider a limit in which we can reliably estimate the amplitude for soliton decay induced by collision with a single W-boson. This amplitude depends on g like $\exp(-cg^{-1/3})$, and is larger than that for spontaneous decay via tunnelling in the same limit. Finally we show that in soliton decays, light $SU(2)_L$ doublet fermions are anomalously produced. Thus we have a calculation of a two body process with energy above the sphaleron barrier in which fermion number is violated.

1   Introduction

In the standard electroweak theory, fermion number violation is present at the quantum level but these processes are seen only outside of ordinary perturbation theory. A baryon number three nucleus can decay into three leptons. The process is described as an instanton mediated tunnelling event[2] leading to an amplitude which is suppressed by $\exp(-8\pi^2/g^2)$, with $g \simeq 0.65$ the $SU(2)$ gauge coupling constant. At energies above the sphaleron barrier[3], fermion number violating processes involving two particles in the initial state are generally believed to be also exponentially suppressed[4]. (At energies comparable to but below the sphaleron barrier, Euclidean

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methods[5] have been used to show that the exponential suppression is less acute than at lower energies, but the approximations used fail at energies of order the barrier height and above.) Unsuppressed fermion number violating processes are generally believed to have of order $4\pi/g^2$ particles in both the initial and final states. This all suggests that fermion number violation will remain unobservable at future accelerators no matter how high the energy, whereas in the high temperature environment of the early universe such processes did play a significant role[6].

In this paper, we explore the robustness of these ideas by studying a variant of the standard model in which the amplitudes for certain fermion number violating collisions, as well as for spontaneous decays, can be reliably estimated for small coupling g. The model is the standard electroweak theory with the Higgs mass taken to infinity and with a Skyrme term[7] added to the Higgs sector. With these modifications, the Higgs sector supports a classically stable soliton which can be interpreted as a particle whose mass is of order the weak scale[8]. Quantum mechanically, the soliton can decay via barrier penetration[9, 10, 11]. Classically, *i.e.,* evolving in Minkowski space using the Euler-Lagrange equations, the soliton can be kicked over the barrier if it is hit with an appropriate gauge field pulse. Correspondingly, the soliton can be induced to decay quantum mechanically if it absorbs the right gauge field quanta. Regardless of whether the decay is spontaneous or induced, ordinary baryon and lepton number are violated in the decay. We shall see that the model has a limit in which fermion number violating amplitudes can be reliably estimated both for processes which occur by tunnelling and for those which occur in two particle collisions between a soliton and a single W-boson with energy above the barrier.

1.1   The Model

To modify the standard model so that it supports solitons, proceed as follows. Note that in the absence of gauge couplings the Higgs sector can be written as a linear sigma model

$$\mathcal{L}_{\mathrm{H}} = \frac{1}{2} \operatorname{Tr} \left[\partial_\mu \Phi^\dagger \partial^\mu \Phi \right] - \frac{\lambda}{4} \left(\operatorname{Tr} \left[\Phi^\dagger \Phi \right] - v^2 \right)^2 \tag{1}$$

where

$$\Phi(\mathbf{x}, t) = \begin{pmatrix} \varphi_0 & -\varphi_1^* \\ \varphi_1 & \varphi_0^* \end{pmatrix} , \tag{2}$$

(φ_0, φ_1) is the Higgs doublet, and $v = 246$ GeV. One advantage of writing the Lagrangian in this form is that it makes the $SU(2)_L \times SU(2)_R$ invariance of the Higgs sector manifest. The scalar field Φ can also be written as

$$\Phi = \sigma U \tag{3}$$

where U is $SU(2)$ valued and σ is a real field. In terms of these variables

$$\mathcal{L}_{\mathrm{H}} = \frac{1}{2}\,\sigma^2\,\mathrm{Tr}\left[\partial_\mu U^\dagger \partial^\mu U\right] + \partial_\mu \sigma \partial^\mu \sigma - \lambda\left(\sigma^2 - \frac{v^2}{2}\right)^2 . \tag{4}$$

The Higgs boson mass is $\sqrt{2\lambda}v$. We work in the limit where the Higgs mass is set to infinity and σ is frozen at its vacuum expectation value $v/\sqrt{2}$. Now

$$\mathcal{L}_H = \frac{v^2}{4}\mathrm{Tr}\left[\partial_\mu U^\dagger \partial^\mu U\right] \tag{5}$$

which is the nonlinear sigma model with scale factor v. We will consider only those configurations for which the fields approach their vacuum values as $|\mathbf{x}| \to \infty$ for all t. We can then impose the boundary condition

$$\lim_{|\mathbf{x}|\to\infty} U(\mathbf{x}, t) = 1 , \tag{6}$$

which means that at any fixed time U is a map from S^3 into $SU(2)$. These maps are characterized by an integer valued winding number which is conserved as the U field evolves continuously. However if we take a localized winding number one configuration and let it evolve according to the classical equations of motion obtained from (5) it will shrink to zero size. To prevent this we follow Skyrme[7] and add a four derivative term to the Lagrangian. The Skyrme term is the unique Lorentz invariant, $SU(2)_L \times SU(2)_R$ invariant term which leads to only second order time derivatives in the equations of motion and contributes positively to the energy.

$$\mathcal{L}_{H\&S} = \frac{v^2}{4}\mathrm{Tr}\left[\partial_\mu U^\dagger \partial^\mu U\right] + \frac{1}{32e^2}\mathrm{Tr}\left[U^\dagger \partial_\mu U \,,\, U^\dagger \partial_\nu U\right]^2 \tag{7}$$

where e is a dimensionless constant.

Of course this Lagrangian is just a scaled up version of the Skyrme Lagrangian which has been used[7, 12, 13] to treat baryons as stable solitons in the nonlinear sigma model theory of pions. To obtain the original Skyrme Lagrangian replace v in (7) by $f_\pi = 93$ MeV. The stable soliton of this theory, the skyrmion, has a mass of $73\,f_\pi/e$ and has a size $\sim 2/(ef_\pi)$[13]. Best fits to a variety of hadron properties give $e = 5.5$[13]. The soliton of (7) has mass $73\,v/e$ and size $\sim 2/(ev)$ and we take e as a free parameter since the particles corresponding to this soliton have not yet been discovered.

The standard electroweak Higgs plus gauge boson sector is obtained by gauging the $SU(2)_L \times U(1)_Y$ subgroup of $SU(2)_L \times SU(2)_R$ in the Lagrangian (1). Throughout this paper we neglect the $U(1)$ interactions. The complete Lagrangian we consider is obtained upon gauging the $SU(2)_L$ symmetry of (7):

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{Tr}F^{\mu\nu}F_{\mu\nu} + \frac{v^2}{4}\,\mathrm{Tr}\left[D^\mu U^\dagger D_\mu U\right] + \frac{1}{32e^2}\,\mathrm{Tr}\left[U^\dagger D_\mu U \,,\, U^\dagger D_\nu U\right]^2 \tag{8}$$

where

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \\
D_\mu U &= (\partial_\mu - iA_\mu)\, U
\end{aligned} \tag{9}$$

with $A_\mu = A_\mu^a \sigma^a/2$ where the σ^a are the Pauli matrices. In the unitary gauge, $U = 1$, and the Lagrangian is

$$\mathcal{L} = \frac{1}{g^2}\left\{ -\frac{1}{2}\operatorname{Tr} F^{\mu\nu} F_{\mu\nu} + m^2 \operatorname{Tr} A_\mu A^\mu + \frac{1}{8\xi}\operatorname{Tr}\, [A_\mu\,,\, A_\nu]^2 \right\}\,, \tag{10}$$

where we have introduced

$$m = \frac{gv}{2} \quad \text{and} \quad \xi = \frac{4e^2}{g^2}\,. \tag{11}$$

Note that the equations of motion derived from (10) agree with those obtained by varying (8) and then setting $U = 1$. Also note that for fixed m and ξ the classical equations of motion are independent of g. Since m is dimensionful and sets the scale, characteristics of the classical theory depend only on the single dimensionless parameter ξ.

1.2   The Soliton and the Sphaleron

The classical lowest energy configuration of (10) has $A_\mu = 0$ and the quantum theory built upon this configuration has three spin-one bosons of equal mass m. In the limit where g goes to zero with e and v fixed (hence, ξ goes to infinity) the Lagrangian (8) is well approximated by its ungauged version (7) which supports a stable soliton, so one suspects that for large ξ the Lagrangian (8) and its gauge-fixed equivalent (10) also support a soliton. In fact, Ambjorn and Rubakov[11] showed that for ξ larger than $\xi^* = 10.35$ the Lagrangian (10) does support a classically stable soliton whereas for $\xi < \xi^*$ such a configuration is unstable. Let $U_1(\mathbf{x})$ be the winding number one soliton, the skyrmion, associated with the ungauged Lagrangian (7). For large ξ, this configuration is a good approximation to the soliton of the gauged Lagrangian (8), so in the unitary gauge the soliton is $A_i^{\text{sol}} \simeq i\,U_1^\dagger \partial_i U_1$, $A_0^{\text{sol}} = 0$. For all $\xi > \xi^*$ the quantum version of the theory described by (10) has, in addition to the three equal mass W-bosons, a tower of particles which arise as quantum excitations about the soliton, just as the proton, neutron and delta can be viewed as quantum excitations about the original skyrmion[12, 13].

The Lagrangian (10) determines a potential energy functional which depends on the configuration $A_\mu(\mathbf{x})$. The absolute minimum of the energy functional is at $A_\mu = 0$. For $\xi > \xi^*$ there is a local minimum at the soliton $A_\mu = A_\mu^{\text{sol}}(\mathbf{x})$ with nonzero energy given by the soliton mass M_{sol}. (Of course a translation or rotation of $A_\mu^{\text{sol}}(\mathbf{x})$ produces

a configuration with the same energy so we imagine identifying these configurations so that the soliton can be viewed as a single point in configuration space.) Consider a path in configuration space from $A_\mu = 0$ to $A_\mu^{\text{sol}}(\mathbf{x})$. The energy functional along this path has a maximum which is greater than the soliton mass. As we vary the path, the maximum varies, and the minimum over all paths of this maximum is a static unstable solution to the classical equations of motion which we call the sphaleron of this theory. (The sphaleron of the standard model[3] marks the lowest point on the barrier separating *vacua* with different winding numbers. Here, the sphaleron barrier separates the vacuum from a soliton with nonzero energy.) For fixed v and e the sphaleron mass M_{sph} goes to infinity as g goes to zero reflecting the fact that for $g = 0$, configurations of different winding (U's with different winding in (7)) cannot be continuously deformed into each other. For fixed g and m, as ξ approaches ξ^* from above the sphaleron mass comes down until at $\xi = \xi^*$ the sphaleron and soliton have equal mass. For $\xi < \xi^*$ the local minimum at nonzero energy has disappeared.

For $\xi > \xi^*$, the classically stable soliton can decay by barrier penetration [9, 10, 11]. This process has been studied in detail by Rubakov, Stern and Tinyakov[14] who computed the action of the Euclidean space solution associated with the tunnelling. They show that in the semi-classical limit as $\xi \to \infty$ the action approaches $8\pi^2/g^2$ whereas as $\xi \to \xi^*$ with g fixed the action goes to zero since the barrier disappears.

1.3   Over the Barrier

In this paper, we focus on processes where there is enough energy to go over the barrier. In the standard model, the sphaleron mass is of order m/g^2 and the sphaleron size is of order $1/m$. This means that for small g two incident W bosons each with energy half the sphaleron mass have wavelengths much shorter than the sphaleron size. This mismatch is the reason that over the barrier processes are generally believed to be exponentially suppressed in $W - W$ collisions. In contrast, in the model we consider we can take a soliton as one of the initial state particles. To the extent that the soliton is close to the sphaleron we have a head start in going over the barrier. We can also choose parameters such that an incident W boson with enough energy to kick the soliton over the barrier has a wavelength comparable to both the soliton and sphaleron sizes.

In Ref. [1], we first look in some detail at solutions to the Minkowski space classical equations of motion derived from the Lagrangian (10). Here, we give only a brief discussion of these solutions and their implications. To simplify the calculations we work in the spatial spherical ansatz[15]. We solve the equations of motion for A_μ in the unitary gauge in the spherical ansatz numerically. We first show how to find the electroweak soliton for any value of $\xi \geq \xi^*$. Then, as initial data we take a single electroweak soliton at rest with a spherical pulse of gauge field, localized at a radius much greater than the soliton size, moving inward toward the soliton. In Ref. [1], we

display one example of a soliton-destroying pulse in detail. At early times we have a soliton and an incident pulse and at late times we have outgoing waves only, the soliton having been destroyed. For ξ within about a factor of two of ξ^*, for all the pulse profiles we have tried with the pulse width comparable to the soliton size, there is a threshold pulse energy above which the soliton is destroyed. The energy threshold is larger than the barrier height, and does depend on the pulse profile.[1] However, the existence of a threshold energy above which the soliton is destroyed seems robust, and in this sense the choice of a particular pulse profile is not important.

A classical wave narrowly peaked at frequency ω with total energy E can be viewed as containing $E/\hbar\omega$ particles. Making a mode decomposition, we can then estimate the number of gauge field quanta, that is W bosons, in a pulse which destroys the electroweak soliton. From (10) we see that such a pulse has an energy proportional to $1/g^2$ for fixed m and ξ. Thus, the particle number N of any such pulse goes like some constant over g^2. For example, at $\xi = 12$ the soliton-destroying pulse presented in detail in Ref. [1] has $g^2 N \sim 2.5$. At this value of ξ, by varying the pulse shape we could reduce $g^2 N$ somewhat but we doubt that we could make it arbitrarily small. Upon reducing ξ towards ξ^* and thus lowering the energy barrier ΔE, smaller values of $g^2 N$ become possible.[2] For example, at $\xi = 11$ we have found pulses with $g^2 N \simeq 1$. In the standard model, finding gauge boson pulses which traverse the sphaleron barrier and which have small $g^2 N$ appears to be much more challenging [16]. Note from the form of (10) that taking g to zero with m and ξ fixed is the semi-classical limit. In this limit, the soliton mass, the sphaleron mass and their difference ΔE all grow as $1/g^2$. The number of particles in any classical pulse which destroys the soliton also grows as $1/g^2$.

The lesson we learn from studying classical solutions is that in the model we are treating, it is straightforward to find soliton destroying, sphaleron crossing, fermion number violating classical solutions. Particular pulse profiles are not required — pulses of any shape we have tried (with sizes comparable to the soliton size) destroy the soliton if their energy is above some shape-dependent threshold.

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[1] We have certainly not found the lowest energy or lowest particle number pulses which destroy the soliton. Indeed, a soliton destroying pulse with energy just above ΔE could be obtained by starting with a slightly perturbed sphaleron, watching it decay to the soliton, and then time reversing. We will see in the next section that for ξ near ξ^* a "pulse" so obtained would be a *very* long train of small amplitude waves, rather than a simple pulse of the kind we have used to destroy solitons in our numerical experiments.

[2] We have worked at values of ξ ranging from 10.5 to 100. There is no reason to go to larger ξ, because as ξ becomes very large the soliton size ($\sim 1/(m\sqrt{\xi})$) becomes much smaller than the sphaleron size ($\sim 1/m$) and the barrier height ΔE grows like $M_{\rm sol}\sqrt{\xi}$. At large ξ, therefore, the energy of soliton destroying pulses must become large compared to $M_{\rm sol}$ and large compared to the inverse sphaleron size. It is nevertheless a logical possibility that such pulses could be found with high frequencies and small values of $g^2 N$. For $\xi = 50$ and above, however, we have only found soliton destroying pulses which have large $g^2 N$. This suggests that because at large ξ the soliton is no longer similar to the sphaleron, we lose the advantage that we have in this model, relative to the standard model, in finding sphaleron crossing solutions with $g^2 N$ small.

The existence of soliton destroying classical pulses has quantum implications beyond a naive estimate of the number of particles associated with a classical wave. In Ref. [1] we give a full and self-contained account of the relationship between classical solutions and the quantum tree approximation in a simple scalar theory. In a theory with an absolutely stable soliton, the Hilbert space of the quantized theory separates into sectors with a fixed number of solitons and states in different sectors have zero overlap[17]. The one soliton sector, for example, is a Fock space of states with one soliton and any number of mesons. The mesons are the quantized fluctuations about the soliton configuration and the states in the one soliton sector are scattering states of mesons in the presence of a soliton. No process, not even one involving large numbers of mesons, connects states in the one soliton sector with states in the vacuum sector. In our theory, the electroweak soliton is not absolutely stable. The Hilbert space has sectors with a fixed number of solitons and any number of W-bosons. However, we argue in Ref. [1] that the existence of classical solutions in which solitons are destroyed demonstrates that there are states in the zero and one soliton sectors of the quantum theory whose overlap in the semi-classical limit is not exponentially small. These states are coherent states with mean number of W-bosons of order $1/g^2$. Knowing that some quantum processes exist which connect the zero and one soliton sectors suggests that we go beyond the semi-classical limit and look for such processes involving only a single incident W-boson.

There is an interesting limit in which we can reliably estimate amplitudes for single particle induced decays. Recall that for m and g fixed, as ξ approaches ξ^* from above the sphaleron mass approaches the soliton mass. We can hold m fixed and pick ξ to be a function of g chosen so that as g goes to zero ξ approaches ξ^* in such a way that $\Delta E = M_{\text{sph}} - M_{\text{sol}}$ remains fixed. We call this the fixed ΔE limit. It is different from the semi-classical limit in that as g goes to zero the classical theory is changing. We will argue in the next section that for ξ near ξ^* it is possible to isolate a mode of oscillation about the soliton whose frequency is near zero, which is in the direction of the sphaleron. This normalizable mode, which we call the λ-mode, is coupled to a continuum of modes with frequencies $\omega > m$. If the λ-mode is sufficiently excited by energy transferred from the continuum modes, then the soliton will decay. We are able to estimate the amplitude for a single W-boson of energy E to excite the λ-mode enough to induce the decay. At threshold the cross section goes like $\exp(-c/g^{1/3})$ where c is a dimensionless constant. In the same limit we can calculate the rate for the soliton to decay by tunnelling and we get $\exp(-(9/(9-2\sqrt{3}))c/g^{1/3})$. Both are exponentially small as g goes to zero and the ratio of the tunnelling rate to the induced decay rate is exponentially small.

1.4   Fermion Production

Having described classical and quantum processes in which electroweak solitons

are destroyed, in Ref. [1] we argue that if we couple a quantized chiral fermion to the gauge and Higgs fields considered in this paper, then soliton destruction implies nonconservation of fermion number. The argument we present treats the gauge and Higgs fields as classical backgrounds. In particular, we ask how many fermions are produced in a background given by a solution to the classical equations of motion in which a soliton is destroyed. We expect that our conclusions will also be valid for soliton destruction induced by a single W-boson.

We introduce fermions into this theory as in the standard electroweak theory but neglecting the $U(1)$ interaction. The left-handed components transform as $SU(2)_L$ doublets whereas the right-handed components are singlets. The fermion mass is generated in a gauge-invariant way by a Yukawa coupling to the Higgs field. For simplicity we only consider the case where both the up and down components of the fermion doublet have equal mass m_f. of Ψ have the same mass m_f. The gauge invariant normal ordered fermion current J_μ is not conserved, that is,

$$\partial_\mu J^\mu = \frac{1}{32\pi^2}\, \epsilon^{\mu\nu\alpha\beta}\, \text{Tr}\, \left(F_{\mu\nu}F_{\alpha\beta}\right)\ . \tag{12}$$

We consider backgrounds given by solutions of the kind found numerically in which an incoming classical pulse destroys the soliton. After the soliton has been destroyed the solution dissipates. By dissipation we mean that at late times the energy density approaches zero uniformly throughout space. This means that at late times the solutions are well described by solutions to the linearized equations of motion

$$\left(\partial_\nu\partial^\nu + m^2\right)A_\mu^{\text{lin}} = 0 \tag{13}$$

in unitary gauge. It is tempting to try to integrate (12) and relate the fermion number violation to the topological charge

$$Q = \frac{1}{32\pi^2}\int d^4x\, \epsilon^{\mu\nu\alpha\beta}\, \text{Tr}\, \left(F_{\mu\nu}F_{\alpha\beta}\right)\ . \tag{14}$$

First, note that because the region of space-time in which $F_{\mu\nu}\neq 0$ is not bounded, there is no reason to expect Q to be an integer. Furthermore, it is shown in Ref. [18] that for a background which satisfies (13) at early and/or late times the integral in (14) is not absolutely convergent and Q cannot sensibly be defined.

In a background given by a solution to the equations of motion which dissipates at early and late times, the number of fermions produced is known to be given by the change in Higgs winding number[18, 19]. In this paper, the Higgs mass is infinite so the Higgs winding number can never change. For solutions with no solitons in the initial or final states, the arguments of Ref. [18] apply, and no fermions are produced. However, if there is a soliton in the initial or final state the assumption of Ref. [18] that the solution dissipates is not satisfied. In Ref. [1], we show that in a background given by a solution in which one soliton is destroyed, one net anti-fermion is produced if the fermion is light ($m_f L \ll 1$ where L is the size of the soliton) and no fermions

are produced if the fermion is heavy ($m_f L \gg 1$). In the $m_f L \gg 1$ case, however, the soliton carries heavy fermion number[20, 10, 21, 1] and when the soliton is destroyed this quantum number is violated. In both cases there is a change of fermion number of minus one and heavy minus light fermion number is conserved as it must be since the heavy and light fermion number currents have the same anomalous divergence.

Suppose we are only interested in light fermion production. We can view the heavy fermion as a device introduced only for the purpose of making an argument. Because we have not included the back reaction of the fermions, heavy or light, on the bosonic background, any conclusions we reach about the light fermion are in fact independent of whether there is or is not a heavy fermion in the theory. Therefore, in any process in which a soliton is destroyed, one net anti-fermion from each light $SU(2)_L$ doublet is anomalously produced.

1.5   Relating the Model to the Real World

The metastable electroweak soliton of the modified Higgs sector is an intriguing object to study. Yet this beast is not found in the standard electroweak theory where the Higgs sector is a linear sigma model with no higher derivative terms. It is reasonable to ask if the modified theory gives a credible description of physics at the weak scale. To date the Higgs boson has not been found. If it is found and the mass is low so that λ of (1) is small then working in the infinite λ limit would not well approximate reality. However, if the Higgs is heavy, then working with infinite λ could be justifiable. Working at the scale v and below, we then integrate out the heavy Higgs, leaving a low energy effective action. In this strongly interacting case, higher derivative terms in the effective action would not be perturbatively small and we would expect all possible higher derivative terms consistent with the symmetries. This effective theory would or would not support stable solitons. If it did then our use of the Skyrme term is justified as a simple way to write an effective action which supports solitons.

It is possible that the Higgs is not fundamental. Rather the Higgs sector may be an effective theory describing the massless degrees of freedom which arise as a result of spontaneous symmetry breaking in some more fundamental theory. For example, this is the basis of technicolor theories in which the symmetry breaking is introduced via a scaled up version of QCD. In technicolor theories one finds technibaryons which can be described as electroweak solitons just as the baryons of QCD can be described as skyrmions. For now, regardless of whether the underlying theory is specifically a technicolor model, as long as we are consistent with symmetry considerations, we are free to choose the effective theory to conveniently describe the particles which interest us. Thus (7) is a simple way to describe three massless bosons (which are eaten in the gauged version (8)) as well as a stable (metastable in (8)) heavy particle. Of course the effective theory includes higher derivative terms other than the Skyrme term, so

it is not the precise form of (8) which we think is plausible, but rather the physical picture which it describes.

It is worth asking what processes can sensibly be described using the effective theory. The effective theory is a derivative expansion in momenta over v. Consider the (fermion number conserving) production of soliton – anti-soliton pairs in $W - W$ collisions. These processes are beyond the regime of applicability of the effective theory because the incident particles have momenta which are greater than v, and the underlying theory must therefore be invoked. (For example, in a technicolor theory the production process would be described as techniquark – anti-techniquark pair production followed by technihadronization.) The effective theory is, however, well-suited to describing soliton decay induced by a single W boson with energy just above ΔE in the fixed ΔE limit. In this limit m is held fixed while $g \to 0$, and thus $v \to \infty$. Therefore, the ratio of the incident W momentum to the scale v is going to zero, and a treatment using the effective theory is justified.

Over the course of this extended introduction, we have sketched all the results presented in full in Ref. [1]. In the remainder of this paper, we give a complete presentation of our treatment of quantum processes in which a single W-boson incident upon the soliton kicks it over the barrier causing it to decay. In Section 3, we do a controlled calculation of this process in a limit in which ξ goes to ξ^* as g goes to zero. In order to do this calculation, however, we first need a better understanding of the classical dynamics of the theory with ξ near ξ^*, and to this we now turn.

2   Classical Dynamics for ξ near ξ^*

In order to discuss the special features of the dynamics of our system for ξ near ξ^*, and because we will need it to discuss the quantum version of this theory, we introduce the Hamiltonian which arises from (10):

$$H = \int \mathrm{d}^3\mathbf{x} \left\{ \frac{1}{g^2} \left[\frac{1}{2}\mathrm{Tr}\, F^{ij} F^{ij} - m^2 \mathrm{Tr}\, A_\mu A^\mu - \frac{1}{8\xi}\mathrm{Tr}\, [A_\mu \,,\, A_\nu]^2 \right] \right.$$
$$\left. + g^2 \mathrm{Tr}\, \Pi^i \Pi^i - 2\mathrm{Tr}\, \left[A_0 D_i \Pi^i \right] \right\} , \tag{15}$$

where

$$\Pi^i = \frac{1}{g^2} F^{i0}$$
$$D_i \Pi^i = \partial_i \Pi^i - i \left[A_i \,,\, \Pi^i \right] . \tag{16}$$

Now A^0 has no conjugate momentum and the A^0 equation is

$$m^2 A^0 + \frac{1}{4\xi} \left[\left[A^i \,,\, A^0 \right] , A_i \right] + g^2 D_i \Pi^i = 0 . \tag{17}$$

This linear equation for A^0 can be solved giving A^0 in terms of A^i and Π^i but we do not need to do this explictly. The Hamiltonian for our system is given by (15) with A^0 determined by (17) and has the general form

$$H = \frac{g^2}{2} \, \Pi \, M^{-1}(A) \, \Pi + \frac{1}{g^2} \, V[A] \; , \tag{18}$$

where the sum over the coordinate $\mathbf{x}$, the spatial index i and the group index are all implicit. The matrix $M^{-1}(A)$ involves derivatives with respect to $\mathbf{x}$, depends on the configuration A, and we assume that $M^{-1}(A)$ is positive. Note that static solutions to the equations of motion, that is those with $\dot\Pi = \dot A = 0$, occur where $\delta V/\delta A = 0$ and have $\Pi = 0$. The classical equation of motion for A which arises from (18) is independent of g. Thus for the discussion of classical dynamics which we are having in this section, we can set $g = 1$. We will restore the g dependence in the next section.

The potential energy functional $V[A]$ has its overall scale set by m but the topography of fixed energy contours is set by ξ. Ambjorn and Rubakov [11] showed that for $\xi > \xi^* = 10.35$ there is a local minimum, the soliton, whereas for $\xi < \xi^*$ this minimum is absent. For $\xi > \xi^*$ there is also a sphaleron, that is a saddle point configuration whose energy is greater than that of the soliton. As ξ approaches ξ^* from above, the sphaleron and soliton merge.

We are particularly interested in configurations which, at least initially, are small perturbations around the soliton. To work with these configurations we find it convenient to make a canonical transformation which has the effect of setting $M^{-1}(A_{\text{sol}}) = 1$ and $\left.\frac{\mathrm{d}M^{-1}}{\mathrm{d}A}\right|_{A_{\text{sol}}} = 0$. To see that this is possible let f_α be some complete set of orthonormal, spatial vector, matrix-valued functions of $\mathbf{x}$, indexed by α, which can be used to expand Π and A. Let the coefficients of the expansion of A relative to the soliton be q^α and the coefficients of the expansion of Π be p_α, that is

$$
\begin{aligned}
A(\mathbf{x}, t) - A^{\text{sol}}(\mathbf{x}) &= \sum_\alpha q^\alpha(t) f_\alpha(\mathbf{x}) \\
\Pi(\mathbf{x}, t) &= \sum_\alpha p_\alpha(t) f_\alpha(\mathbf{x}) \; .
\end{aligned} \tag{19}
$$

(Note that the transformation from $A(\mathbf{x}, t)$, $\Pi(\mathbf{x}, t)$ to $q^\alpha(t)$, $p_\alpha(t)$ is canonical.) Upon making this transformation, (18) has the form

$$H = \frac{1}{2} \, g^{\alpha\beta}(q) p_\alpha p_\beta + V(q) \; . \tag{20}$$

A canonical transformation of the form

$$q'^\alpha = q'^\alpha(q) \quad \text{and} \quad p'_\alpha = \frac{\partial q^\beta}{\partial q'^\alpha} p_\beta \tag{21}$$

can be viewed as a general coordinate transformation with p_α transforming as a covariant vector. It is always possible to choose coordinates such that

$$g'^{\alpha\beta} = \frac{\partial q'^\alpha}{\partial q^\delta} \frac{\partial q'^\beta}{\partial q^\epsilon} g^{\delta\epsilon} \tag{22}$$

is equal to $\delta^{\alpha\beta}$ with $\partial g'^{\alpha\beta}/\partial q'^\epsilon = 0$ at any given point. In fact this can be accomplished at $q^\alpha = 0$ (the soliton) with a transformation of the form $q'^\alpha = C^\alpha_\beta q^\beta + C^\alpha_{\beta\delta} q^\beta q^\delta$. This means that the Hamiltonian (20) can be written as

$$H = \frac{1}{2} p_\alpha \left[\delta^{\alpha\beta} + \mathcal{O}(q^2) \right] p_\beta + V(q) \ , \tag{23}$$

where we have made the required canonical transformation and dropped the primes. Note that $V(q=0) = M_{\text{sol}}$ and $(\partial V/\partial q^\alpha)|_{q=0} = 0$.

For $\xi > \xi^*$ consider small oscillations about the soliton. The frequencies squared are given by the eigenvalues of the fluctuation matrix $\partial^2 V/\partial q^\alpha \partial q^\beta$ at $q = 0$. The soliton is a localized object so fluctuations far from the soliton propagate freely. Therefore the fluctuation matrix at the soliton has a continuous spectrum above m^2. A given soliton configuration and a translation or rotation of that configuration have the same energy and both solve $\partial V/\partial q^\alpha = 0$. This implies that at $q = 0$ there are six zero eigenvalues of $\partial^2 V/\partial q^\alpha \partial q^\beta$. The associated modes which correspond to translating and rotating the soliton are not of interest to us and will be systematically ignored.

For ξ close to ξ^* we now argue that there is one normalizable mode whose frequency ω_0 goes to zero as ξ goes to ξ^*. To see this we write

$$\frac{\partial V}{\partial q^\alpha}\bigg|_{q_{\text{sph}}} = \frac{\partial V}{\partial q^\alpha}\bigg|_{q=0} + \frac{\partial^2 V}{\partial q^\alpha \partial q^\beta}\bigg|_{q=0} q^\beta_{\text{sph}} + \frac{1}{2} \frac{\partial^3 V}{\partial q^\alpha \partial q^\beta \partial q^\epsilon}\bigg|_{q=0} q^\beta_{\text{sph}} q^\epsilon_{\text{sph}} + \ldots \ . \tag{24}$$

At the soliton ($q = 0$) and at the sphaleron the first derivatives are zero. As ξ approaches ξ^* the sphaleron and soliton merge so q^α_{sph} goes to zero. It is useful to introduce the normalized function $\bar{q}_{\text{sph}}$

$$\bar{q}^\alpha_{\text{sph}} = \frac{q^\alpha_{\text{sph}}}{Q} \tag{25}$$

where

$$Q^2 = \sum_\alpha q^\alpha_{\text{sph}} q^\alpha_{\text{sph}} \ . \tag{26}$$

As ξ goes to ξ^*, Q goes to zero but $\bar{q}_{\text{sph}}$ does not. From (24) we then have

$$\frac{\partial^2 V}{\partial q^\alpha \partial q^\beta}\bigg|_{q=0} \bar{q}^\alpha_{\text{sph}} \bar{q}^\beta_{\text{sph}} = -\frac{1}{2} Q \frac{\partial^3 V}{\partial q^\alpha \partial q^\beta \partial q^\epsilon}\bigg|_{q=0} \bar{q}^\alpha_{\text{sph}} \bar{q}^\beta_{\text{sph}} \bar{q}^\epsilon_{\text{sph}} + \mathcal{O}(Q^2) \ . \tag{27}$$

For $\xi > \xi^*$ the fluctuation matrix $\partial^2 V/\partial q^\alpha \partial q^\beta$ at the soliton has only positive eigenvalues (except for the translation and rotation zero modes which play no role in this discussion). Equation (27) tells us that at $\xi = \xi^*$ where $Q = 0$, the fluctuation matrix has a zero eigenvalue with eigenvector $\bar{q}_{\mathrm{sph}}$ whereas for ξ close to ξ^* there is a small eigenvalue, ω_0^2, whose associated eigenvector is close to $\bar{q}_{\mathrm{sph}}$. Note that $\bar{q}_{\mathrm{sph}}$ points from the soliton to the sphaleron. Thus the low frequency mode, which we call the λ-mode, is an oscillation about the soliton close to the direction of the sphaleron.

For $\xi > \xi^*$, at the sphaleron there is one negative mode, that is one negative eigenvalue of the appropriately defined fluctuation matrix. As ξ comes down to ξ^* the sphaleron and soliton become the same configuration so this negative eigenvalue must come up to zero in order for the spectra of the fluctuation matrices of the soliton and sphaleron to agree at $\xi = \xi^*$. Therefore for ξ close to ξ^* the unstable direction off the sphaleron has a small negative curvature. There are two directions down from the sphaleron. One heads toward the soliton and the other heads (ultimately) to the classical vacuum at $A = 0$. We see that for ξ near ξ^* the soliton can be destroyed by imparting enough energy to the λ-mode since it is this mode which is pointed towards the sphaleron and beyond.

We wish to describe the interaction of the λ-mode with the other degrees of freedom. We use the Hamiltonian written in the form (23). At this point it is convenient to make an orthogonal transformation on the $\{q^\alpha\}$ so that the transformed set are the eigenvectors of the soliton fluctuation matrix $\partial^2 V/\partial q^\alpha \partial q^\beta|_{q=0}$. We will label these vectors as q_ω where ω^2 is the eigenvalue of the fluctuation matrix. The eigenfunctions include:

i) The continuum states q_ω with eigenvalues $\omega^2 > m^2$. (Note that for each ω^2, in general, there is more than one eigenvector. The extra labels on q_ω are suppressed in our compact notation.)

ii) The normalizable state $q_{\omega_0} \equiv \lambda$ with eigenvalue ω_0^2 which goes to zero as ξ goes to ξ^*.

iii) The zero eigenvalue states associated with translation and rotation.

iv) Other normalizable states which might exist but whose frequencies do not have any reason to approach zero as ξ goes to ξ^*.

Up to cubic order the Hamiltonian (23) is

$$
\begin{aligned}
H = M_{\mathrm{sol}} \ &+ \ \frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 \lambda^2 + \frac{b}{3}\lambda^3 \\
&+ \ \frac{1}{2}\int_m \mathrm{d}\omega\, p_\omega^2 + \frac{1}{2}\int_m \mathrm{d}\omega\, \omega^2\, q_\omega^2 + \int_m \mathrm{d}\omega \mathrm{d}\omega' \mathrm{d}\omega''\, c(\omega,\omega',\omega'')\, q_\omega q_{\omega'} q_{\omega''} \\
&+ \ \lambda^2 \int_m \mathrm{d}\omega\, d(\omega) q_\omega + \lambda \int_m \mathrm{d}\omega \mathrm{d}\omega'\, e(\omega,\omega') q_\omega q_{\omega'} + \dots
\end{aligned}
\tag{28}
$$

where in the ellipses we now include all terms with modes of type iii) and iv) as well as higher order interactions of the λ-mode and the continuum modes. p is the momentum conjugate to λ and p_ω is the momentum conjugate to q_ω. The number b and the functions c, d and e are determined by the soliton configuration. For example $d(\omega)$ is presumably peaked at values of ω which correspond to wavelengths of order the size of the soliton. As ξ goes to ξ^* we know that ω_0 goes to zero but we expect no dramatic behavior of b, c, d or e in this limit.

Consider the λ-mode potential

$$V(\lambda) = \frac{1}{2}\omega_0^2\lambda^2 + \frac{b}{3}\lambda^3 + \dots \ . \tag{29}$$

There is a local minimum at $\lambda = 0$ which is the soliton and a local maximum at $\lambda = -\omega_0^2/b$, which is approximately the sphaleron, where the second derivative is $-\omega_0^2$. We work with ξ sufficiently close to ξ^* that ω_0 is small. This means that λ at the sphaleron is small and if we only study dynamics up to and just beyond the sphaleron we are justified in neglecting the quartic and higher terms in λ. We also see that as ξ goes to ξ^* so that ω_0 goes to zero, the soliton and sphaleron come together and at $\xi = \xi^*$ the λ potential has an inflection point at $\lambda = 0$ and the soliton is no longer classically stable.

In order to discover the relationship between ω_0 and $(\xi - \xi^*)$ as ξ approaches ξ^*, it is necessary to study the behavior of the λ-mode potential as ξ approaches ξ^*. In (29) for every value of ξ, we have shifted λ so that the minimum of the potential is at $\lambda = 0$. This ξ dependent change of variables obscures the behavior of the coefficients of the potential before the shift. Calling the unshifted variable $\bar{\lambda}$, then if we expand the potential in terms of $\epsilon \equiv \xi - \xi^*$ about $\epsilon = 0$ where there is an inflection point, we have

$$V(\bar{\lambda}, \epsilon) = \mathcal{O}(\epsilon)\bar{\lambda} + \mathcal{O}(\epsilon)\bar{\lambda}^2 + \left(\bar{b} + \mathcal{O}(\epsilon)\right)\bar{\lambda}^3 + \dots \ , \tag{30}$$

where $\bar{b}$ is a constant. We know that the coefficients of $\bar{\lambda}$ and $\bar{\lambda}^2$ are zero at $\epsilon = 0$, and we assume that these coefficients can be expanded about $\epsilon = 0$ and we know of no reason for the order ϵ terms to vanish. For $\epsilon > 0$ the minimum of the potential is at $\bar{\lambda} \sim \epsilon^{1/2}$, ($\lambda$ is shifted relative to $\bar{\lambda}$ by this amount), and at the minimum of the potential $\partial^2 V/\partial\bar{\lambda}^2 \sim \epsilon^{1/2}$, that is

$$\omega_0^2 \sim (\xi - \xi^*)^{1/2} \ . \tag{31}$$

A small amplitude oscillation of the λ mode will decay because of its coupling to the continuum modes which can carry energy away from the soliton. However for $\omega_0 < m$ this decay is very slow in the sense that the characteristic time for the decay is much greater than $1/\omega_0$. To understand this consider $\lambda(t)$ as a source for radiation in the continuum via the coupling $\lambda^2 \int_m d\omega d(\omega) q_\omega$ in the Hamiltonian (28). Suppose that $\lambda(t)$ is a purely sinusoidal oscillation with frequency ω_0 and with an amplitude which is small. Radiation with frequency ω_0 is not possible because the continuum

frequencies begin at $\omega = m$. However, λ^2 has frequency $2\omega_0$ and therefore if $\omega_0 > m/2$ the coupling will excite propagating modes with $\omega = 2\omega_0$ and the λ oscillation will radiate at twice its fundamental frequency. Because the coupling is of order λ^2, the rate of energy loss will be small. If $\omega_0 < m/2$ then radiation at $\omega = 2\omega_0$ is also not possible. However, if $m/3 < \omega_0 < m/2$ the $\lambda^3 q_\omega$ coupling (which we have not written in (28) because it is fourth order) allows the λ oscillation to radiate at three times its fundamental frequency. There is another source of radiation with $\omega = 3\omega_0$. The potential for the λ-mode is not exactly quadratic so the λ oscillation, although periodic, is not exactly sinusoidal. If the period of the oscillation is $2\pi/\omega_0$, λ will be a sum of terms of the form $\sin \omega_0 t$, $\sin 2\omega_0 t$, $\sin 3\omega_0 t$,... with diminishing coefficients. This means that λ^2 will also be a sum of terms of this form. Those terms in λ^2 with frequencies greater than m will excite radiation via the $\lambda^2 q_\omega$ coupling. As ω_0 is reduced from m toward zero, the radiation is produced only by higher order couplings and by higher harmonics, and therefore the amplitude is reduced and the decay takes longer.

We have numerical evidence for this behavior within the spherical ansatz. In unitary gauge in the spherical ansatz, A_μ is written in terms of four real functions of r and t. Gauss' law specifies one of these functions in terms of the other three, so that a solution to the equations of motion is fully specified by the real functions $\rho(r, t)$, $\theta(r, t)$ and $a_1(r, t)$ defined in Ref. [1]. To watch an oscillating soliton radiate for a long time, we implement energy absorbing boundary conditions at the large r boundary of the simulation lattice, as described in Ref. [1]. We wish to excite the λ-mode and watch it oscillate. We describe in Ref. [1] a convenient way of choosing a configuration which is a soliton plus a small perturbation where the perturbation is preferentially in modes with lower frequencies. We then use this configuration as the initial condition for the equations of motion. The resulting evolution is shown in Figures 1 and 2 for $\xi = 10.4$. The functions ρ, θ, and a_1 (we show ρ only) all oscillate about the value they take at the soliton and the period of oscillation is $16.69\,m^{-1}$. We identify this with the λ-mode and so obtain $\omega_0 = 0.3764\,m$. Furthermore we see that away from the soliton there is a small amplitude train of outgoing radiation. After a brief initial period during which any perturbations not in the λ-mode radiate away, the outgoing radiation settles down to a frequency $1.129\,m$, three times the fundamental frequency. (At $r = 10\,m^{-1}$, we see in Figure 2 that the frequency $3\omega_0$ oscillation of ρ has a small modulation with frequency ω_0. This is the tail of the λ-mode oscillation and is not seen at larger values of r.) The radiation causes the amplitude of the λ-mode to decrease very slowly — by about 4% over 80 oscillations. We have done similar simulations at $\xi = 11$ and $\xi = 12$ also, where we find $\omega_0 = 0.80\,m$ and $\omega_0 = 0.98\,m$ respectively. In these simulations, the oscillating soliton emits radiation with $\omega = 2\omega_0$, and the amplitude of the radiation and the rate of decay of the fundamental oscillation are larger than in Figure 2. The values of ω_0 for $\xi = 10.4$, 11, and 12 which we have found numerically are in good agreement with the relationship (31). This numerical evidence suggests that we are justified in using the Hamiltonian (28) to describe the
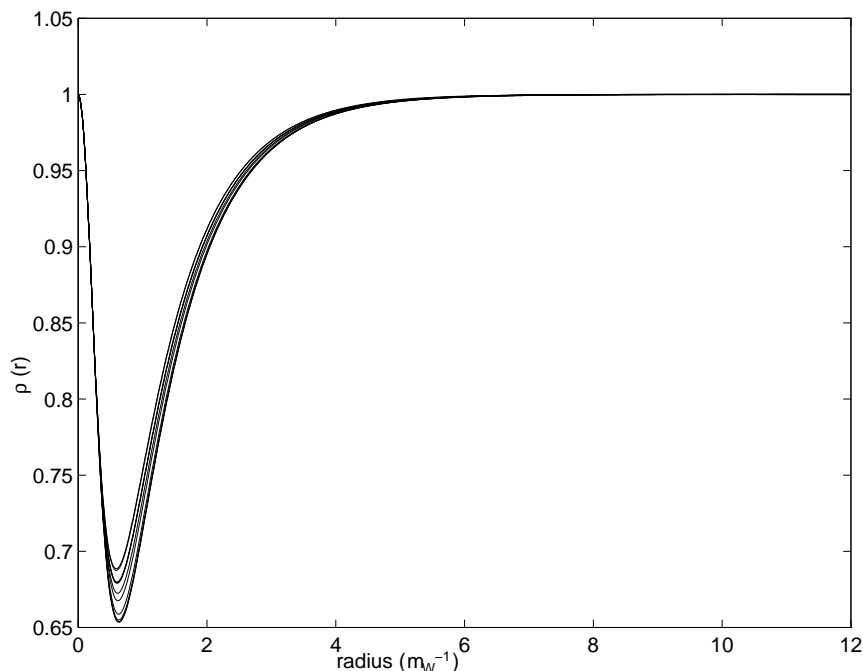
Figure 1: As described in the text, we have perturbed the $\xi = 10.4$ soliton and let it evolve for a long time. Here, we show $\rho(r)$ for a series of different times: $t = 0$, 144, 288, ... 1440 m^{-1}. This shows the envelope of the oscillation of ρ. In Figure 2, we show ρ at $r = 0.608/m$ and $r = 10/m$ as a function of time.

long-lived normalizable λ-mode with $\omega_0 < m$ and its coupling to the continuum. In the next section we will quantize this Hamiltonian and use it to describe the excitation of the λ-mode by single W-boson quanta.

Finally we note that in principle it is possible to destroy a soliton with a minimum energy pulse, *i.e.* one whose energy is just above ΔE, and for ξ close to ξ^* this energy is small. To find the form of this pulse we could time reverse a solution which starts at the sphaleron and is given a gentle push towards the soliton. For ξ close to ξ^* so that the λ-mode has a small frequency, the configuration takes a very long time to settle down to the soliton and in the process emits a very long train of low amplitude outgoing waves. Although the time-reversed solution consisting of a very long train of incoming low amplitude waves being absorbed by the soliton would eventually go over the sphaleron barrier and result in soliton decay, it would be rather difficult to set up initial conditions which produce this complicated, finely tuned, incoming configuration. Thus, the minimum energy soliton destroying pulses are not easy to build although we have seen that with some extra energy, for ξ near ξ^*, the soliton is easily killed.
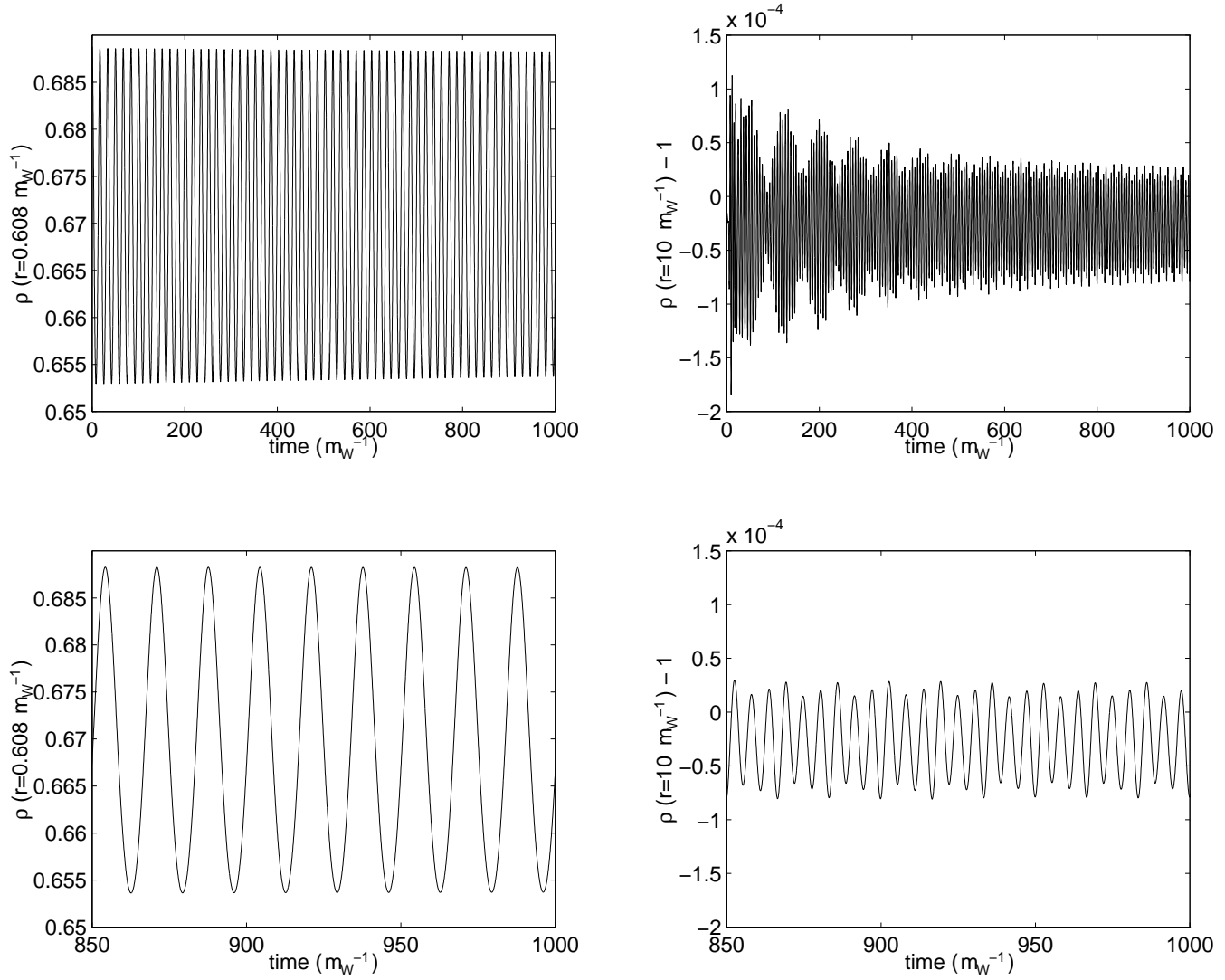
Figure 2: In the left panels, we show ρ at $r = 0.608/m$ as a function of time. It oscillates with period $16.69/m$, and the amplitude of the oscillation is decreasing very slowly. In the right panels, we show ρ at $r = 10/m$, to display the outgoing travelling waves shed by the oscillating soliton. These waves have three times the frequency of the fundamental oscillation seen at $r = 0.608/m$. Note that the amplitude of the outgoing waves is so small that they are invisible in the plots of $\rho(r)$ on the preceding page. We conclude that for $\xi = 10.4$ the soliton has an almost stable mode of oscillation with frequency $\omega_0 = 0.374\,m$ — the λ-mode — which slowly radiates waves with frequency $3\,\omega_0$.

3   Quantum Processes in the Fixed ΔE Limit

In the previous section we saw that for ξ close to ξ^* it is possible to identify a low frequency vibration of the soliton, the λ-mode, with frequency ω_0 much less than m. If enough energy is transferred to this mode the soliton will decay. In this section we discuss the quantum mechanics of this mode. In this quantum setting the soliton can decay by barrier penetration as well as by being kicked over the barrier by a single W-boson. We will see that if we work in a limit where ΔE is held fixed as we take g to zero, then we can reliably estimate the leading terms in both the tunnelling and induced decay rates.

The Hamiltonian for just the λ-mode coming from (28) is given by

$$H_\lambda = \frac{g^2}{2}\, p^2 + \frac{1}{g^2}\, \left\{ \frac{1}{2}\omega_0^2\lambda^2 - \frac{b}{3}\lambda^3 + \dots \right\} \tag{32}$$

where we have restored the g dependence. Note that ω_0, b and all the terms in the ellipses depend on ξ and m but not on g. We have changed the sign of λ for later convenience. As ξ goes to ξ^*, ω_0 goes to zero but the other terms are presumed not to change much. The classical soliton is at $\lambda = 0$ while the sphaleron is at $\lambda = \omega_0^2/b$ from which we have

$$\Delta E = \frac{1}{6}\, \frac{\omega_0^6}{g^2 b^2}\ . \tag{33}$$

The fixed ΔE limit has g going to zero with ξ taken to ξ^* in such a way that (33) is fixed. Since $b(\xi, m)$ does not vary much as ξ goes to ξ^*, we see that in this limit $\omega_0 \sim g^{1/3}$. Using (33) and (31), we see that $g^2\Delta E \sim (\xi - \xi^*)^{3/2}$ so that in order to take the fixed ΔE limit we take g to zero with $(\xi - \xi^*) \sim g^{4/3}$. (The reader who is concerned that the coefficient of λ^2 in (32), ω_0^2/g^2, goes to infinity in the fixed ΔE limit should note that because of the g^2 in front of the p^2 in (32) the frequency of oscillation is ω_0.)

When taking the fixed ΔE limit, it proves convenient to rescale according to

$$\begin{aligned} \lambda' &= \lambda\,\omega_0/g \sim \lambda\, g^{-2/3} \\ p' &= p\,g/\omega_0 \sim p\, g^{2/3} \\ b' &= b\,g/\omega_0^3 \sim b\, g^0\ . \end{aligned} \tag{34}$$

Writing the Hamiltonian (32) in terms of the new variables and then dropping the primes we obtain

$$H_\lambda = \omega_0^2\, \frac{p^2}{2} + V(\lambda)\ , \tag{35}$$

where

$$V(\lambda) = \frac{1}{2}\lambda^2 - \frac{b}{3}\lambda^3 + \dots\ . \tag{36}$$

After rescaling, the sphaleron is at $\lambda = 1/b$ and the barrier height is given by

$$\Delta E = 1/(6b^2) \ . \tag{37}$$

Quartic and higher terms in $V(\lambda)$ are all suppressed by powers of $g/\omega_0 \sim g^{2/3}$. Note that ω_0 now plays the role of $\hbar$ in the Hamiltonian (35). As g goes to zero in the fixed ΔE limit, ω_0 goes to zero like $g^{1/3}$ and a semi-classical (WKB) treatment is appropriate in order to compute the leading small-g behavior of the soliton destruction cross-section.

In the fixed ΔE limit, the ground state of the quantum soliton has the λ degree of freedom in a wave function $\psi_0(\lambda)$ which is described approximately by a harmonic oscillator ground state wave function:

$$\psi_0(\lambda) \sim \left(\frac{1}{\pi\omega_0}\right)^{1/4} \exp\left(-\frac{\lambda^2}{2\omega_0}\right) \ . \tag{38}$$

There are three relevant scales in λ, which differ in their g-dependence. First, the width of the ground state wave function $\sqrt{\langle\psi_0|\lambda^2|\psi_0\rangle}$ goes like $\sqrt{\omega_0} \sim g^{1/6}$. The second scale, which goes like g^0, is the distance in λ between the sphaleron at $\lambda = 1/b$ and the minimum at $\lambda = 0$. Note also that (38) is a good approximation to ψ_0 for λ such that the cubic term in $V(\lambda)$ can be neglected relative to the quadratic term, namely for $|\lambda| \ll 1/b$. Finally, note that the quartic and higher terms in $V(\lambda)$ can be neglected for λ less than of order $\omega_0/g \sim g^{-2/3}$, the third scale. Hence, as g is taken to zero with ΔE fixed, truncating the potential at cubic order becomes valid for larger and larger λ.

The soliton will decay if the λ degree of freedom tunnels under the barrier given by the potential $V(\lambda)$ shown in Figure 3. The rate is of the form

$$\Gamma = Ce^{-2B} \tag{39}$$

where the factor B is

$$B = \frac{\sqrt{2}}{\omega_0} \int_0^{3/2b} \mathrm{d}\lambda\sqrt{\lambda^2/2 - b\lambda^3/3} = \frac{3}{5}\frac{1}{\omega_0 b^2} = \frac{18}{5}\frac{\Delta E}{\omega_0} \ . \tag{40}$$

We are able to neglect the width of the wave function (38) in this calculation because as g goes to zero it is small compared to the change in λ during the tunnelling process. Since in the fixed ΔE limit $\omega_0 \sim g^{1/3}$ we see that the tunnelling rate goes as $\exp(-\mathrm{constant}/g^{1/3})$. For the approximation to be reliable we require that B be much greater than one. This in turn requires that g be small.

We can compare this calculation with that of Rubakov, Stern and Tinyakov[14] who numerically calculated the action of the Euclidean space solution which tunnels under the barrier. They used the equations of motion of the full $3+1$ dimensional theory with the restriction to the spherical ansatz. At $\xi = 12$ we have $\Delta E = 1.2\, m/g^2$,
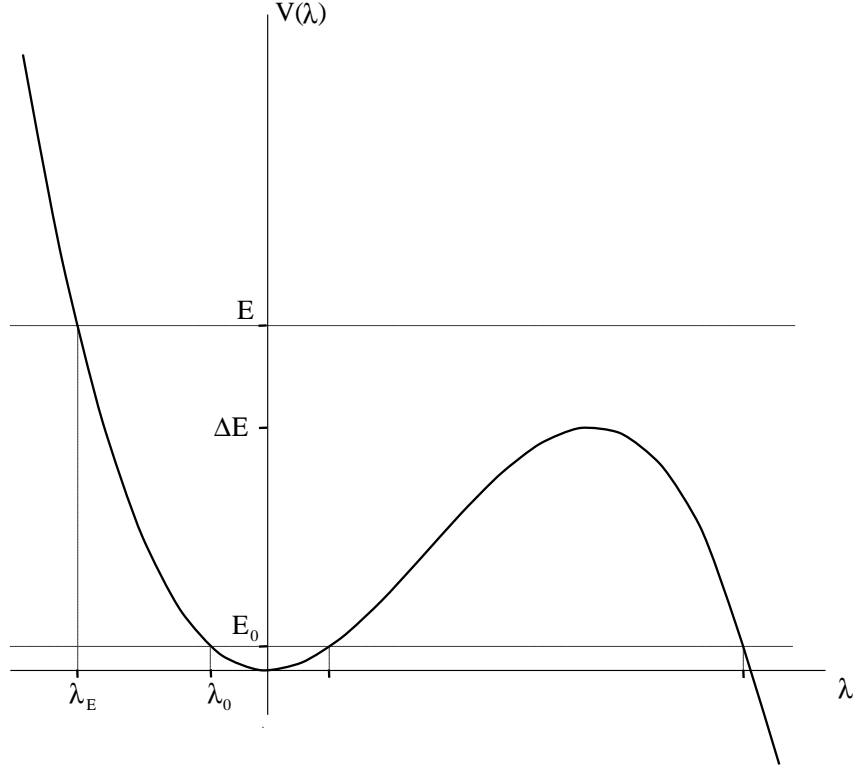
Figure 3: The potential $V(\lambda)$ for real λ. For later use, the energies E_0 and E are also shown. ψ_0 has three turning points, and $\lambda = \lambda_0$ is the left-most of the three. ψ_E has one turning point at $\lambda = \lambda_E$.

$\omega_0 = 0.98\, m$ giving $g^2 B = 4.4$ which is to be compared with what we read off Figure 2 of Ref. [14], namely $g^2 B = 4 \pm 1$. This agreement again supports the view that the λ mode is the relevant degree of freedom for discussing soliton decay for ξ near ξ^*.

We now turn to induced soliton decay. Our picture is that the soliton will decay if the λ-mode is excited to a state with energy above ΔE. The λ-mode couples to the continuum modes q_ω which can bring energy from afar to the soliton. The free quantum Hamiltonian for the q_ω is

$$
\begin{aligned}
H_{q_\omega} &= \frac{1}{2} \int_m d\omega \left[g^2 p_\omega^2 + \frac{\omega^2}{g^2} q_\omega^2 \right] \\
&= \int_m d\omega\, \omega \left[a_\omega^\dagger a_\omega + 1/2 \right]
\end{aligned}
\tag{41}
$$

where

$$
a_\omega = \frac{1}{\sqrt{2\omega}} \left(\frac{\omega q_\omega}{g} + i g p_\omega \right) \ .
\tag{42}
$$

The q_ω have been chosen to diagonalize the fluctuation matrix at the soliton. Therefore H_{q_ω} describes non-interacting massive W-bosons propagating in a fixed soliton

background. For each value of ω there are actually an infinite number of different W-boson quanta. For example there are the states with frequency ω and all values of angular momentum relative to the soliton center. These extra labels are omitted throughout but their presence is understood.

The λ-mode couples to the continuum modes through cubic couplings of the form

$$H_{\text{int}} = \frac{1}{\omega_0^2} \left\{ \lambda^2 \int_m d\omega \, d(\omega) \, q_\omega + \frac{\omega_0}{g} \lambda \int_m d\omega \, d\omega' \, e(\omega, \omega') \, q_\omega q_{\omega'} \right\} \qquad (43)$$

which appear in (28). We have rescaled λ according to (34). The couplings (43) arose upon expanding about the soliton. The functions $d(\omega)$ and $e(\omega, \omega')$ are peaked at values of ω corresponding to wavelengths of order the size of the soliton. They are also only peaked if the unspecified labels allow large overlap with the soliton. For example even with ω chosen so that $(\omega^2 - m^2)^{-1/2} \sim$ soliton size, it is only the low partial waves which have $d(\omega)$ and $e(\omega, \omega')$ large.

The first term in (43) allows for the absorption of a single W-boson by the soliton. The W-boson energy E is transferred to the λ-mode. The second term in (43) allows a single W-boson to scatter inelastically off the soliton, transferring energy E to the λ-mode. We now calculate the rate for the absorption process; the calculation for the scattering process is similar. (The coefficients of the λ and λ^2 operators have different g-dependence, but this will not affect the leading g-dependence of the cross-section for either process.) Assuming that the soliton starts in its ground state, in order for the soliton to decay we require $E + \omega_0/2 > \Delta E$. Since $\omega_0 \ll \Delta E$ we can approximate this as $E > \Delta E$. In the fixed ΔE limit we are free to choose ΔE to be a constant times m where the constant is of order unity. (Recall that m is held fixed throughout this paper.) Now the soliton size is roughly $2/(m\sqrt{\xi})$ and in the fixed ΔE limit ξ goes to $\xi^* = 10.35$. Thus the W-boson wavelength and the soliton size can be comparable. There is no length scale mismatch and $d(E)$ need not be small.

Using Fermi's Golden Rule we now calculate the cross section for $W + \text{soliton} \to$ anything with no soliton. Let $|\mathbf{k}\rangle$ be a single W-boson state with energy E, normalized to unit particle flux. Now

$$\langle 0|H_{\text{int}}|\mathbf{k}\rangle = \frac{\lambda^2}{\omega_0^2} \int_m d\omega \, d(\omega) \, \langle 0|q_\omega|\mathbf{k}\rangle \equiv g \frac{\lambda^2}{\omega_0^2} \, \bar{d}(\mathbf{k}) \;, \qquad (44)$$

where we have defined $\bar{d}(\mathbf{k})$ so that it is independent of g (see (42)). The λ-mode starts in the state $\psi_0(\lambda)$ with energy $\sim \omega_0/2$ which again we neglect relative to ΔE. The interaction (44) can cause a transition to a state $\psi_E(\lambda)$ in which the λ-mode has energy E. Since the width of ψ_0 is $\sim g^{1/6} \ll 1$, it is tempting to try approximating the states with $E > \Delta E$ as plane waves

$$\psi_E(\lambda) \sim \frac{1}{\omega_0^{1/2} E^{1/4}} \, \exp\left(i\sqrt{2E}\lambda/\omega_0\right) \;. \qquad (45)$$

The cross section for a transition from ψ_0 to ψ_E is

$$\sigma_{\text{destruction}} = \mathcal{N} \left(\frac{g \, \bar{d}(\mathbf{k})}{\omega_0^2} \right)^2 \mathcal{I}(E)^2 \ , \tag{46}$$

where $\mathcal{N}$ is a g-independent constant and where $\mathcal{I}(E)$ is the integral

$$\mathcal{I}(E) = \int d\lambda \, \psi_0(\lambda) \, \lambda^2 \, \psi_E(\lambda) \ . \tag{47}$$

If we take ψ_0 and ψ_E as in (38) and (45) respectively, $\mathcal{I}(E)$ is easily evaluated, yielding

$$\mathcal{I}(E) \sim \exp\left(-E/\omega_0\right) \ , \tag{48}$$

where we have dropped all prefactors. This result is in fact incorrect.[3] While it is true that (38) and (45) yield a good approximation to the integrand where the integrand is biggest, the result (48) is exponentially smaller than the integrand. This raises the possibility that corrections to the wave functions neglected to this point may change (48). We must, therefore, use WKB wave functions which take into account the quadratic and cubic terms in the potential $V(\lambda)$. As $g \to 0$ in the fixed ΔE limit, $\omega_0 \to 0$ and using semi-classical wave functions becomes a better and better approximation. We show below that for $E = \Delta E$ the leading dependence of the of $\mathcal{I}(E)$ as $g \to 0$ in the fixed ΔE limit is in fact that of (48) with the coefficient of $\Delta E/\omega_0$ being $(18 - 4\sqrt{3})/5$ instead of 1. Thus, we will find that even though the soliton destruction process does not involve tunnelling, the correct cross-section is exponentially small as $\omega_0 \sim g^{1/3}$ goes to zero. The reader who is not interested in the details of the evaluation of $\mathcal{I}(E)$ can safely skip to equation (60).

We now wish to evaluate the leading semi-classical dependence of

$$\mathcal{I}(E_0, E) = \int d\lambda \, \psi_{E_0} \, \lambda^2 \, \psi_E \tag{49}$$

in the fixed ΔE limit where $E > \Delta E$ and $\Delta E > E_0 > 0$ and where ψ_E and ψ_{E_0} are WKB wave functions for the Hamiltonian (35). See Figure 3. The reader may be concerned that (49) is infinite. (Both wave functions are real, and for large positive λ the integrand (49) has a non-oscillatory piece which grows like $\lambda^2 \lambda^{-3/2}$.) However, when the relevant limits are taken correctly, the answer we seek is in fact finite. Recall that our problem reduces to that of the λ mode in a cubic potential only for $|\lambda| < \omega_0/g \sim g^{-2/3}$. Therefore, we should do the λ integration from $\lambda = -\Lambda$ to $\lambda = +\Lambda$, where Λ is real and positive and where we take Λ to infinity more slowly than $g^{-2/3}$ as g goes to zero. The result of such an evaluation would go like $\Lambda^{3/2} \exp(-\text{constant}/\omega_0)$. Because we do not take Λ to infinity before taking g to zero, the prefactor does not make the result infinite.

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[3]We are grateful to D. T. Son for noticing this, and for pointing us toward the correct answer.
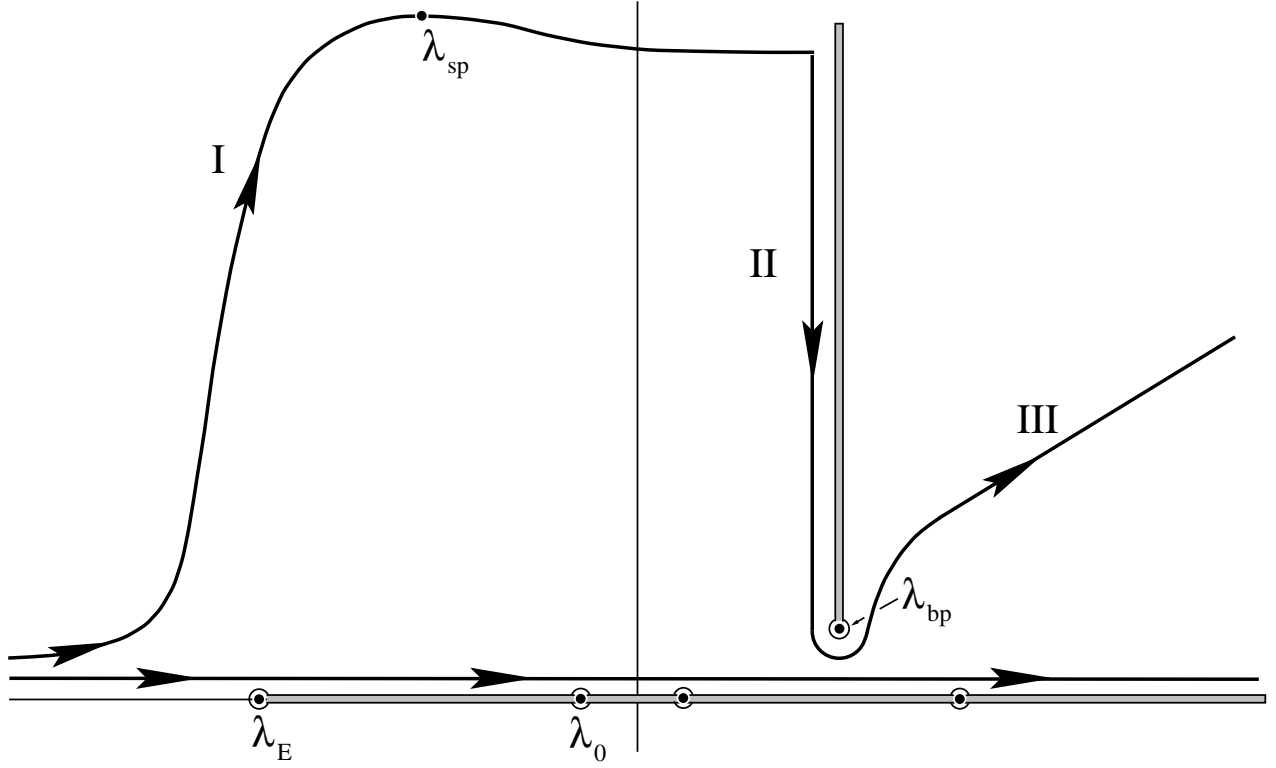
Figure 4: This figure is a sketch showing the important points in the complex λ-plane discussed in the text. The branch points are marked with dots, and the branch cuts are shaded. The contour in (5.19), just above the real axis, and the deformed contour we use to evaluate the integral are both shown.

The evaluation of matrix elements of operators between semi-classical states has been treated by Landau[22], and although his final answer does not apply to our problem, we follow his method to its penultimate step. Landau's method yields only the leading (*i.e.* exponential) dependence of such matrix elements, and says nothing about the prefactors. Thus, using Landau's method yields the leading small-g dependence of (49) irrespective of whether the prefactors make the integral infinite. In the calculation which follows, it nevertheless proves convenient to multiply the integrand in (49) by $\exp(-J\lambda^2/\omega_0)$ with J a constant. This does in fact render the integral finite, but it may also modify the exponential dependence of the result. Therefore, after the $g \to 0$ limit has been taken we must take the $J \to 0$ limit. Landau's method[22] applied to our problem yields

$$
\begin{aligned}
\mathcal{I}(E_0, E) \sim \quad & \mathrm{Im}\Bigg\{ \int d\lambda \, \frac{\omega_0 \, \lambda^2}{[(V(\lambda) - E_0)\,(V(\lambda) - E)]^{1/4}} \\
\times \quad & \exp\left[\frac{1}{\omega_0}\left(\int_{\lambda_0}^{\lambda} dx \sqrt{2\,(V(x) - E_0)} - \int_{\lambda_E}^{\lambda} dx \, \sqrt{2\,(V(x) - E)} - J\lambda^2\right)\right]\Bigg\}
\end{aligned} \tag{50}
$$

In this expression, λ is treated as complex and it is understood that the contour has been deformed into the upper half plane. This is done both in order to avoid the turning points on the real axis shown in Figure 4, and because in deriving (50) Landau uses expressions for WKB wave functions which are valid only in the upper half plane and not on the real axis. The first square root in the exponent in (50) is taken to be positive on the real axis for $\lambda < \lambda_0$ and the second is taken to be positive on the real axis for $\lambda < \lambda_E$.

The equation $V(x) - E = 0$ has three roots. One is at λ_E, on the negative real axis, and the other two, at $\lambda_{\rm bp}$ and $\lambda_{\rm bp}^*$, have nonzero imaginary parts. (For $E \to \Delta E$, $\lambda_{\rm bp}$ goes to the real axis at $\lambda_{\rm sph} = 1/b$.) In evaluating (50) we must keep in mind that at $\lambda = \lambda_{\rm bp}$ in the upper half plane, the integrand has a branch point. This singularity will play an important role in our analysis. (Unlike in the example treated explicitly by Landau, it does not arise from a singularity in $V(\lambda)$.) The branch cut from $\lambda_{\rm bp}$ must not cross the real axis, and it is convenient to take it to run upward vertically. The integrand in (50) is a function which is analytic in the upper half plane except at $\lambda_{\rm bp}$ and along the associated cut. To evaluate the integral, we are free to push the contour upward away from the real axis as long as we ensure that it does not touch the branch point $\lambda_{\rm bp}$ or cross the branch cut.

We now evaluate the leading exponential dependence of (50).[4] To this end, we drop the prefactors in (50). We write the integral as

$$\int d\lambda \, \exp \frac{1}{\omega_0} \Big[X(\lambda) + iY(\lambda) \Big] \tag{51}$$

where X and Y are real and where

$$X + iY = \int_{\lambda_0}^{\lambda} dx \, \sqrt{2 \left(V(x) - E_0 \right)} - \int_{\lambda_E}^{\lambda} dx \, \sqrt{2 \left(V(x) - E \right)} - J\lambda^2 \ . \tag{52}$$

It is easy to check that for $J = 0$ the integrand in (51) has no saddle points at finite λ. However, making J nonzero introduces a saddle point at large $|\lambda|$ which moves off to infinity as J is taken to zero and it is convenient for us to evaluate the integral with nonzero J and then take the $J \to 0$ limit.

We now describe the behavior of X at large $|\lambda|$. Write $\lambda = \Lambda \exp i\theta$. We have chosen the branch cut to run vertically and so it is at $\theta = \pi/2$ for large Λ. To the right of the cut, that is for $\pi/2 > \theta > 0$, as Λ goes to infinity

$$X \sim -\Lambda^{5/2} \sin(5\theta/2) - J\Lambda^2 \cos(2\theta) \ , \tag{53}$$

and the J term is subleading. X goes to $+\infty$ at large Λ for $\pi/2 > \theta > 2\pi/5$ and goes to $-\infty$ for $2\pi/5 > \theta > 0$. The descent to $-\infty$ is most rapid for $\theta = \pi/5$. To the left of the cut, that is for $\pi \geq \theta \geq \pi/2$, as Λ goes to infinity

$$X \sim X^* + \Lambda^{-1/2} \sin(\theta/2) - J\Lambda^2 \cos(2\theta) \ , \tag{54}$$

[4]The analysis described below and the result (60) were provided by A. V. Matytsin.

where X^* is a constant independent of J, Λ, and θ. (For $J = 0$, as Λ goes to infinity for $\pi \geq \theta \geq \pi/2$, $X \to X^*$ and $Y \to 0$.) For nonzero J, there is a saddle point at finite λ. For small J, this saddle point is at $\theta \simeq 3\pi/5$ and $\Lambda \sim J^{-2/5}$. Thus, as $J \to 0$ the saddle point recedes to infinity as promised, and $J\Lambda^2$ at the saddle point goes to zero. Therefore, in the $J \to 0$ limit X at the saddle point goes to the value X^*.

We now deform the contour as sketched in Figure 4. For nonzero J, the saddle point is at finite λ and we choose the contour to follow the path of steepest descent from this saddle point. To the left of the saddle point, the steepest descent path curves toward the real axis, and then approaches the real axis asymptotically. As we discuss below, $X(\lambda_{\mathrm{bp}})$ is *greater* than X^*. Therefore, to the right of the saddle point, the path of steepest descent from the saddle point cannot get around the branch point and necessarily runs into the branch cut. After reaching the cut, the next section of the path *ascends* as it traverses (II), following the cut inward toward the origin, until it reaches the region of the branch point λ_{bp}. Along (II), X ascends monotonically from X^* to $X(\lambda_{\mathrm{bp}})$. Y is not constant. Then, to the right of the cut, the contour follows the path of steepest descent (III) toward infinity along $\theta = \pi/5$.

There are two contributions to the integral (51). First, the saddle point makes a contribution which goes like $\exp(X^*/\omega_0)$. (Note that we take the $g \to 0$ limit and then take the $J \to 0$ limit.) The second contribution arises because the path must ascend from the saddle point at infinity as it traverses (II) in order to get around the branch point, before then descending along (III) to the right. Therefore, the integral (51) receives a contribution from the region of the branch point which goes like $\exp(X(\lambda_{\mathrm{bp}})/\omega_0)$. In sum, therefore, the integral (50) goes like

$$\mathcal{I}(E_0, E) \sim \exp(X^*/\omega_0) + \exp(X(\lambda_{\mathrm{bp}})/\omega_0) \,, \tag{55}$$

where we have dropped the prefactors, about which Landau's method says nothing. At this point, we can take the $E_0 \to 0$ and $E \to \Delta E$ limits simply by setting $E_0 = 0$ and $E = \Delta E$. Prior to this point in the calculation, taking these limits would require careful treatment of branch points. Henceforth we set $E_0 = 0$ and compute $\mathcal{I}(E) = \mathcal{I}(0, E)$.

It only remains to evaluate the relative size of $X(\lambda_{\mathrm{bp}})$ and X^*. Both $X(\lambda_{\mathrm{bp}})$ and X^* depend on E. After some calculation one finds that for $E = \Delta E$

$$X^* = -\omega_0 B = -\frac{18}{5}\Delta E \,, \tag{56}$$

where B is the tunnelling amplitude computed in (40), and

$$X(\lambda_{\mathrm{bp}}) = -\sqrt{2}\int_0^{1/b} d\lambda\sqrt{\lambda^2/2 - b\lambda^3/3} = -\frac{1}{5b^2}\left(3 - \frac{2}{\sqrt{3}}\right) = -\frac{18}{5}\Delta E\left(1 - \frac{2}{3\sqrt{3}}\right) \,, \tag{57}$$

so $X(\lambda_{\mathrm{bp}})$ is the larger (*i.e.* least negative) of the two at $E = \Delta E$. At large E, both $X(\lambda_{\mathrm{bp}})$ and X^* decrease like $-E^{5/6}$. For $E > \Delta E$, the integrals in (52) must

be evaluated numerically. We find that both $X(\lambda_{\rm bp})$ and X^* decrease monotonically with increasing energy, and $X(\lambda_{\rm bp})$ is always greater than X^*. Consequently, the integral is dominated by the region of the branch point for all energies $E \geq \Delta E$. That is,

$$\mathcal{I}(E) \sim \exp(X(\lambda_{\rm bp})/\omega_0) \tag{58}$$

and

$$\sigma_{\rm destruction} \sim \bar{d}^2({\rm k}) \exp(2X(\lambda_{\rm bp})/\omega_0) \ , \tag{59}$$

where we have dropped all prefactors except $\bar{d}$. Thus, although the integrand has a saddle point (at infinity), the integral is not dominated by that saddle point. This occurs because the path of steepest descent from the saddle point necessarily runs into the branch cut. Equivalently, the presence of the branch cut prevents the actual contour of integration from being deformed into a path of steepest descent through the saddle point. Although the path can be deformed to pass through the saddle, it must ascend from the saddle to the region of the branch point. (Note that although $X(\lambda_{\rm bp}) > X^*$ for all energies $E \geq \Delta E$, $X(\lambda_{\rm bp})$ is greater than $B\omega_0$, and the rate for induced soliton decay is greater than the tunnelling rate, only for E within a range of energies which we determine numerically to be $\Delta E \leq E \lesssim 1.74\Delta E$.)

Because $X(\lambda_{\rm bp})$ decreases monotonically with increasing E, the cross section (59) for the soliton to be destroyed by a single W-boson is least suppressed by $\mathcal{I}(E)$ at threshold. For $E = \Delta E$ the soliton destruction cross section goes like

$$\sigma_{\rm destruction} \sim \bar{d}^2({\rm k}) \exp\left(-\frac{36 - 8\sqrt{3}}{5}\Delta E/\omega_0\right) \tag{60}$$

as $g \to 0$ in the fixed ΔE limit.

We expect $d(E)$ and accordingly $\bar{d}(\mathbf{k})$ to be appreciable when $E \sim \Delta E$ so long as ΔE is comparable to the inverse soliton size, which is of order the inverse W-mass. Under these conditions, there will be no length scale mismatch and $d(E)$ will not depend sensitively on E for $E \sim \Delta E$, so $\sigma_{\rm destruction}$ will be maximized for $E = \Delta E$. Thus the maximum rate for soliton decay induced by collision with a single W-boson is proportional to $\exp(-(36/5 - 8\sqrt{3}/5)\Delta E/\omega_0)$. This is to be compared with the tunnelling rate in the same limit which is proportional to $\exp(-(36/5)\Delta E/\omega_0)$. Both go to zero as g goes to zero like $\exp(-{\rm constant}/g^{1/3})$, but the ratio of the tunnelling rate to the induced decay rate is exponentially small.

We have computed the cross section for a single W-boson to be absorbed by the soliton and to excite the λ-mode to a continuum state above the barrier, which in our picture results in soliton decay. The cross section for a W-boson to destroy the soliton by scattering off the soliton and transferring energy E to the λ-mode can be calculated using the second term in (43). The calculation is similar to the one we have done and the result has the same exponential factor as in (60) but would have a different prefactor. Because the exponent in (60) includes $\omega_0^{-1} \sim g^{-1/3}$, these cross sections go to zero faster than any power of g as g goes to zero in the fixed ΔE limit.

Note that this suppression arises even though the process does not involve tunnelling and even though there is no length scale mismatch. It arises as a consequence of the limit in which we have done the computation, because in that limit destroying the soliton reduces to exciting a single degree of freedom to an energy level infinitely many ($\sim \Delta E/\omega_0$) levels above its ground state. Thus, taking $g \to 0$ at fixed ΔE makes the computation tractable but makes the induced decay rate exponentially small, albeit larger than the tunnelling rate.

4　Concluding Remarks

We have described a theory which agrees with the standard electroweak model at presently accessible energies but which includes a metastable soliton with mass of order several TeV. This Higgs sector soliton may have a dual description as a bound state particle made of more fundamental constituents or it may be that the Higgs sector is fundamental and when quantum effects are taken into account, a metastable soliton is found. In any event, given the soliton, under certain circumstances we can reliably estimate the rate for collision induced decays. The parameters of the theory can be chosen so that the soliton configuration is close to the sphaleron configuration, which means that using the soliton as an initial particle makes it easy to find sphaleron crossing processes. Indeed, we have found classical solutions in which the soliton is destroyed where the incoming pulse corresponds to a quantum coherent state with $\sim 1/g^2$ W-bosons. The rate for such processes is not exponentially suppressed as g goes to zero. Furthermore in the limit g goes to zero with $\Delta E = M_{\mathrm{sph}} - M_{\mathrm{sol}}$ fixed we can reliably estimate the rate for a two particle scattering process in which a single incident W-boson kicks the soliton over the barrier causing it to decay. We have argued that in all processes in which the soliton disappears fermion number is violated. This model may be relevant only as a theoretical foil, as a demonstration that fermion number violating high energy scattering processes can be very different than in the standard model. However if no light Higgs boson is discovered, it is even possible that Nature may be described by such a model.

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